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Advection-nonlinear-diffusion model of flare accelerated electron transport in Type III solar radio bursts

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ABSTRACT

Electrons accelerated by solar flares and observed as type III solar radio bursts are not only a crucial diagnostic tool for understanding electron transport in the inner heliosphere but also a possible early indication of potentially hazardous space weather events. The electron beams travelling in the solar corona and heliosphere along magnetic field lines generate Langmuir waves and quasilinearly relax towards a plateau in velocity space. The relaxation of the electron beam over the short distance in contrast to large beam-travel distances observed is often referred to as Sturrok's dilemma. Here, we develop a new electron transport model with quasilinear distance/time self-consistently changing in space and time. The model results in a nonlinear advection-diffusion equation for the electron beam density with nonlinear diffusion term that inversely proportional to the beam density. The solution predicts slow super-diffusive (ballistic) spatial expansion of a fast propagating electron beam. The model also provides the evolution of the spectral energy density of Langmuir waves, which determines brightness temperature of plasma radiation in solar bursts. The model solution is consistent with the results of numerical simulation using kinetic equations and can explain some characteristics of type III solar radio bursts.

1. INTRODUCTION

The signatures of accelerated electrons in solar flares are observed over a wide range of frequencies from low radio frequencies in the interplanetary space to gamma-ray range at the Sun. Hard X-ray and radio observations provide the most direct signatures of electron acceleration and propagation in the solar atmosphere and in the interplanetary space (see, e.g. Lin 1985; Holman et al. 2011; Benz 2017, as reviews). Since the early radio X-ray and in-situ electron observations (Lin 1974) a close association between X-rays and type III solar radio bursts has been noted (e.g. Lin 1985; Krucker et al. 2007; Reid et al. 2014) suggesting that the cloud of non-thermal beam electrons travels from the solar flare site into the interplanetary space. These energetic electrons and associated type III emission could be used for the forecasting of radiation hazards from solar energetic ion events (Posner 2007). However, the quantitative description of electron transport responsible for type III bursts is a long-standing challenge. The solar flare electrons propagating along open magnetic field lines are believed to be responsible for type III solar radio bursts via generation of Langmuir waves and subsequent conversion of these Langmuir waves into escaping radio emission (Ginzburg & Zhelezniakov 1958). Because the faster electrons overtake the slower ones, the conditions for beam-plasma instability quickly appear leading to a plateau in velocity space and Langmuir waves generation. The characteristic time of kinetic beam-plasma instability (quasilinear relaxation) is normally short $\tau_{\rm q} \approx n_p/(n_b\omega_{pe})$, where $\omega_{pe}^2 = 4\pi e^2 n_p/m$ is the electron plasma frequency, n_b, n_p are the electron densities of beam and plasma respectively (Vedenov et al. 1961). The mean free path for 30 keV electrons is $\lambda_{\rm q} = v\tau_{\rm q} \sim 100$ km ($\tau_{\rm q} \sim 10^{-3}$ s.) for typical type III beam parameters $n_b/n_p \sim 10^{-5}$, $\omega_{pe}/2\pi = 100$ MHz, $v \sim 10^{10}$ cm/s in the solar corona. Fast quasilinear relaxation produces beam deceleration (Sturrock 1964, see also (Kaplan & Tsytovich 1973; Muschietti 1990; Karlicky 1997; Yoon et al. 2012; Timofeev et al. 2015; Akbari et al. 2021; Krafft & Savoini 2023)), a problem that has become known as Sturrock's dilemma. Indeed, spacecraft observations show solar flare energetic electrons are accompanied by type III solar radio bursts (Lin 1970; Fainberg & Stone 1970). There are two broad approaches to resolve the dilemma: one avenue invokes modification to quasilinear Langmuir wave generation (e.g. Papadopoulos et al. (1974); Bardwell & Goldman (1976); Sauer et al. (2019) attributed the solution to Sturrock's dilemma to nonlinear effects introduced by the oscillating two-stream instability, stabilization by plasma density inhomogeneities (e.g. Goldman & Dubois 1982; Muschietti et al. 1985), cyclic Langmuir collapse (Che et al. 2017)), while the second avenue highlights that electron beam is spatially nonuniform, so Langmuir waves would preferentially be generated at the front of the beam and absorbed at the back (e.g. Zheleznyakov & Zaitsev 1970; Zaitsev et al. 1972; Mel'nik 1995; Mel'nik & Kontar 2000). Numerical solutions of kinetic equations (Takakura & Shibahashi 1976; Magelssen & Smith 1977; Grognard 1982; Takakura 1982; Kontar 2001a; Hannah et al. 2009; Li et al. 2008; Reid & Kontar 2013; Ratcliffe et al. 2014) broadly support the latter: although quasilinear relaxation flattens the electron distribution, spatial inhomogeneity of the electron beam allows electrons to propagate large distances. While the time-consuming numerical simulations provide important insights, analytic theory is essential to relate observable properties of type III bursts and electron beam properties.

One major simplification for the challenge is to utilise the smallness of quasilinear time in comparison to the characteristic time of the beam and to seek the hydrodynamic description at the timescales larger than the quasilinear relaxation (e.g. Ryutov & Sagdeev 1970; Mel'nik 1995). This is a good assumption due to the smallness of the characteristic time of beam-plasma interaction (the quasilinear time, $\tau_{\rm q}$) compared to the characteristic time of the beam t, $\tau_{\rm q} \ll t$. However, due to finite size of the electron beam, the quasilinear time being inversely proportional to the beam density should change from small (high beam density near beam center) to large (small beam density away from the beam center) values. Furthermore, the parameter $\tau_{\rm q}/t$ becomes a function of space and time through dependency on the beam density.

In the present work we address this theoretical challenge noting that the quasilinear time has to be explicitly treated as a function of time and space depending on the electron number density of the electron beam. Thus, the relaxation process is not going to be taking place at the same rate in all points of space. This non-linear description of fast diffusion naturally leads to a more realistic analytical solutions, comparable to the results of the numerical simulations shown in the literature.

2. KINETIC DESCRIPTION OF ELECTRONS AND LANGMUIR WAVES

Quasilinear theory describes the propagation of electrons along magnetic field lines in a weakly magnetized plasma, and the resonant interaction of the electrons with Langmuir waves e.g. $\omega_{\rm pe} = kv$, where $\omega_{\rm pe}$ is the local plasma frequency, k is the wave number and v is the velocity. The quasilinear equations (Vedenov & Velikhov 1963; Drummond & Pines 1964), provide kinetic description for electrons and Langmuir waves in type III solar radio bursts. As the electrons follow magnetic field lines, the reduced field-aligned electron distribution function $f(v,x,t) = \int f(\mathbf{v})d\mathbf{v}_\perp$ and the spectral energy density of Langmuir waves $W(v,x,t) = \int W(\mathbf{k})d\mathbf{k}_\perp$ evolve following the non-linearly coupled kinetic equations:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{4\pi^2 e^2}{m^2}\frac{\partial}{\partial v}\frac{W}{v}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}D\frac{\partial f}{\partial v}\,, \tag{1}$$

$$\frac{\partial W}{\partial t} = \frac{\pi\omega_{pe}}{n_p}v^2 W\frac{\partial f}{\partial v}\,, \tag{2}$$

where $\int W dk = U$ and $\int f {\rm d}v = n_b$ are the energy density of Langmuir waves and the number density of the electron beam. For completeness, we note that the spontaneous terms are not taken into account in the kinetic model (equations 1,2), since the beam-driven level of Langmuir waves is much higher than the spontaneous/thermal one (see discussion in Lyubchyk et al. 2017). Equation (2) does not include the spatial transfer of the energy by Langmuir waves, since the group velocity of Langmuir waves is small ($v_{gr} \sim v_{T_e}^2/v \ll v$, where v_{T_e} is the electron thermal velocity). The kinetic equations (1-2) do not have an analytical solution and additional assumptions are required to solve this system of equations. For completeness, we note that the equations (1,2) are coupled to nonlinear processes responsible for decay/coalescence of Langmuir waves. The wave-wave interactions are normally treated numerically, see e.g. the large-scale simulations by Ratcliffe et al. (2014).

3. HYDRODYNAMIC DESCRIPTION

The characteristic time of beam-plasma interaction is normally small $\tau_{\rm q} \ll t = d/v$, where d is the size of an electron beam. The smallness of quasilinear time allows using hydrodynamic description for beam electrons and Langmuir waves (Ryutov & Sagdeev 1970; Mel'nik 1995; Mel'nik et al. 1999; Ryutov 2018), so the electron distribution function $f(v,x,t)$ in the kinetic equations (1,2) is the series in small parameter $\tau_{\rm q} v/d$

$$f = f^0 + f^1 + ...\,, \tag{3}$$

which is conceptually similar to the Chapman-Enskog theory of a neutral gas dominated by collisions (Chapman & Cowling 1970). Unlike the Chapman-Enskog theory, this theory utilises fast beam-plasma interaction via Langmuir waves. Substituting the expansion (3) into kinetic equations (1,2), we have in $0^{\rm th}$-order or the fastest terms when $\tau_{\rm q}v/d \to 0$:

$$0 = \frac{4\pi^2 e^2}{m^2}\frac{\partial}{\partial v}\frac{W^0}{v}\frac{\partial f^0}{\partial v} \propto \tau_q^{-1}, \tag{4}$$

$$0 = \frac{\pi\omega_{pe}}{n_p} v^2 W^0 \frac{\partial f^0}{\partial v} \propto \tau_{\rm q}^{-1}. \tag{5}$$

and hence dominant for $d/v \gg \tau_{\rm q}$. This leads to a well-known result that the $0^{\rm th}$-order solution is a plateau in the velocity space since $\partial f^0/\partial v = 0$ (e.g. Vedenov et al. 1967)

$$f^0\left(v,x,t\right) = \begin{cases} p\left(x,t\right), & 0 < v < u(x,t) \\ 0, & v \geq u(x,t) \end{cases} \tag{6}$$

and an enhanced level of Langmuir waves, so that the spectral energy density of Langmuir waves becomes

$$W^0\left(v,x,t\right) = \begin{cases} W_0\left(v,x,t\right), & 0 < v < u(x,t) \\ 0, & v \geq u(x,t) \end{cases} \tag{7}$$

where zero-order terms f^0 and W^0 turn the right-hand sides of kinetic equations (1, 2) to zero. In other words, any initially unstable electron distribution function relaxes to a plateau and Langmuir waves are generated within quasilinear time $\sim \tau_{\rm q}$. Here $p\left(x,t\right)$ is the plateau height and $u(x,t)$ is the maximum electron velocity, so that the number density of electrons is

$$n(x,t) = \int_0^{u(x,t)} p\left(x,t\right) {\rm d}v = p\left(x,t\right) u(x,t). \tag{8}$$

Following (Mel'nik et al. 1999), one can find the equations for $p(x,t)$, $u(x,t)$ and $W_0(v,x,t)$. Integrating equation (1) over v from $v = 0$ to $v = u(x,t)$, one obtains the equation for electron number density $n(x,t) = p(x,t)u(x,t)$

$$\frac{\partial pu}{\partial t} + \frac{1}{2}\frac{\partial pu^2}{\partial x} = \frac{\partial n}{\partial t} + \frac{1}{2}\frac{\partial nu}{\partial x} = 0, \tag{9}$$

which is the hydrodynamic continuity equation or conservation of electrons. Integrating equation (1) over v between $u - \xi$ and $v = u + \xi$, with $\xi \to 0$, gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{10}$$

while combining equations (1,2) one obtains

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = \frac{\omega_{pe}}{m}\frac{\partial}{\partial v}\frac{1}{v^3}\frac{\partial W_0}{\partial t}\,, \tag{11}$$

which is the equation for the spectral energy density of Langmuir waves. Equation (11) can be integrated to find a solution for a initial value problem. Following Mel'nik (1995); Mel'nik et al. (1999), the equations for $p(x,t)$, $u(x,t)$ and $W(v,x,t)$ can be integrated for given initial conditions. For initial condition

$$f(v,x,t=0) = n_b \exp(-x^2/d^2) g(v), \tag{12}$$

where n_b is the electron beam density at $x = 0$, and $g(v) = 2v/v_0^2$ for $v < v_0$, the solution of equations (9, 10,11) gives (see (e.g. Kontar 2001b))

$$u(x,t) = v_0\,, \tag{13}$$

$$p(x,t) = \frac{n_b}{v_0}\exp(-(x - v_0 t/2)^2/d^2)\,, \tag{14}$$

$$W_0(v,x,t) = \frac{m}{\omega_{pe}} v^4 \left(1 - \frac{v}{v_0}\right) p(x,t)\,, \tag{15}$$

The solution (13-15) suggests a beam-Langmuir-wave structure, i.e. electron beam together with Langmuir waves propagate with speed $v_0/2$ preserving the initial size d. The equations as well as solution assume that the relaxation proceeds at the same rate for all x and t, which is evidently not true due to finite spatial size of the electron beam d. The electron number density is higher near the peak of the beam-plasma structure and decreases away. Therefore, the relaxation of electrons should proceed at different rate $\tau_{\rm q}^{-1} \propto n(x,t)$ in various spatial locations and one should take into account the spatial variation of $\tau_{\rm q}$ due to variation of electron number density $n(x,t)$ of the beam.

4. HYDRODYNAMICS WITH NON-LINEAR DIFFUSION

To address the inhomogeneity of quasilinear time, we retain the f^1 term in the expansion of f and substituting (3) into (1), one finds

$$\frac{\partial\left(f^0+f^1\right)}{\partial t}+v\frac{\partial\left(f^0+f^1\right)}{\partial x}=\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial x}+\frac{\partial f^1}{\partial t}+v\frac{\partial f^1}{\partial x}=\frac{\partial}{\partial v}D\frac{\partial f^1}{\partial v}\,, \tag{16}$$

where the first order terms are retained. Further, integrating this equation over velocity from 0 to ∞ gives

$$\frac{\partial}{\partial t}\int\limits_0^{v_0}\left(f^0+f^1\right)\mathrm{d}v+\frac{\partial}{\partial x}\int\limits_0^{v_0}v\left(f^0+f^1\right)\mathrm{d}v=\left.D\frac{\partial f^1}{\partial v}\right|_0^{v_0}\,, \tag{17}$$

where due to the quasilinear relaxation at $t \gg \tau_{\rm q}$, a plateau is considered to be established in the electron distribution function i.e. $f^0(v,x,t)=p(x,t)$ for $v<v_0$. Hence, equation (17) becomes

$$\frac{\partial p v_0}{\partial t}+\frac{1}{2}\frac{\partial p v_0^2}{\partial x}+\frac{\partial}{\partial t}\int\limits_0^{v_0}f^1\mathrm{d}v+\frac{\partial}{\partial x}\int\limits_0^{v_0}vf^1\mathrm{d}v=0\,, \tag{18}$$

where $\int\limits_0^{v_0}f^1\mathrm{d}v=0$ because $\int\limits_0^{v_0}(f^0+f^1)\mathrm{d}v=n(x,t)$. Since the electron number density can be written $n(x,t)=p(x,t)v_0$, we can write equation (18) as

$$\frac{\partial n}{\partial t}+\frac{v_0}{2}\frac{\partial n}{\partial x}+\frac{\partial}{\partial x}\int\limits_0^{v_0}vf^1\mathrm{d}v=0\,, \tag{19}$$

where $f^1(v,x,t)$ is to be found. The procedure to find f^1 is similar to the derivation of a spatial diffusion coefficient from pitch-angle scattering diffusion coefficient (Jokipii 1966; Hasselmann & Wibberenz 1970; Schlickeiser 1989). Multiplying equation (16) by v_0 and subtracting equation (19) one finds

$$\left(v-\frac{v_0}{2}\right)\frac{\partial n}{\partial x}+v_0\frac{\partial f^1}{\partial t}+v_0 v\frac{\partial f^1}{\partial x}-\frac{\partial}{\partial x}\int\limits_0^{v_0}vf^1dV=v_0\frac{\partial}{\partial v}D\frac{\partial f^1}{\partial v}\,, \tag{20}$$

and retaining only zero order terms, one finds equation for $f^1(v,x,t)$

$$\frac{1}{v_0}\left(v-\frac{v_0}{2}\right)\frac{\partial n}{\partial x}=\frac{\partial}{\partial v}D\frac{\partial f^1}{\partial v}\,, \tag{21}$$

where the right-hand side is zero order due to fast plateau formation (equation 4). Since the quasilinear relaxation operates $0<v<v_0$, velocity diffusion coefficient $D=\pi\omega_{pe}^2/(mn_p)(W/v)$ [One can see this explicitly from W_0 solution (15)] should be zero at the boundary velocities, i.e.:

$$\left.D\right|_{v=0}=\left.D\right|_{v=v_0}=0. \tag{22}$$

These boundary conditions allow us to integrate equation (21) over v and obtain

$$D\frac{\partial f^1}{\partial v}=\frac{1}{v_0}\frac{\partial n}{\partial x}\int\limits_0^{v}\left(v'-\frac{v_0}{2}\right)\mathrm{d}v'=\frac{1}{v_0}\frac{\partial n}{\partial x}\frac{1}{2}v\left(v-v_0\right)+C_1\,, \tag{23}$$

where $C_1 = 0$ due to $D|_0 = D|_{v_0} = 0$, yielding

$$\frac{\partial f^1}{\partial v} = \frac{v\,(v - v_0)}{2v_0 D}\frac{\partial n}{\partial x}. \tag{24}$$

Further, integrating equation (24) over v, we find expression for f^1

$$f^1 = \frac{1}{2v_0}\frac{\partial n}{\partial x}\int_0^v \frac{v'^2 - v_0 v'}{D}\mathrm{d}v' + C_2\,, \tag{25}$$

where the constant C_2 is determined from $\int_0^{v_0} f^1 \mathrm{d}v = 0$, (see Appendix A for details). Therefore, f^1 becomes

$$f^1 = \frac{1}{2v_0}\frac{\partial n}{\partial x}\left[\int_0^v \frac{v'^2 - v_0 v'}{D}\mathrm{d}v' - \frac{1}{v_0}\int_0^{v_0}(v_0 - v')\,\frac{v'^2 - v_0 v'}{D}\mathrm{d}v'\right]\,, \tag{26}$$

and from equation (A2)

$$\int_0^{v_0} v f^1 \mathrm{d}v = -\frac{1}{4v_0}\frac{\partial n}{\partial x}\int_0^{v_0}\frac{v^2\,(v_0 - v)^2}{D}\mathrm{d}v\,. \tag{27}$$

Hence, the transport equation for electron number density (equation 19) takes the form

$$\frac{\partial n}{\partial t} + \frac{v_0}{2}\frac{\partial n}{\partial x} = -\frac{\partial}{\partial x}\int_0^{v_0} v f^1 \mathrm{d}v = \frac{\partial}{\partial x}\frac{1}{4v_0}\frac{\partial n}{\partial x}\int_0^{v_0}\frac{v^2\,(v_0 - v)^2}{D}\mathrm{d}v\,, \tag{28}$$

which is the modified equation of particle conservation (compare to equation 9).

4.1. *Advection and non-linear diffusion*

The velocity diffusion coefficient $D \propto W$ is determined by the level of Langmuir waves. Taking the spectral energy density of Langmuir waves W^0 given by equation (15), one can write for D:

$$D = \pi\frac{\omega_{pe}}{v_0}\frac{n(x,t)}{n_p}v^3\left(1 - \frac{v}{v_0}\right) = D_0 v^3\left(1 - \frac{v}{v_0}\right)\,, \tag{29}$$

where $D_0 = \pi\frac{\omega_{pe}}{v_0}\frac{n(x,t)}{n_p}$. Finally substituting (29) into (28) and integrating from $v_{\min}$ (instead of 0 as in equation 28) leads us to advection diffusion equation

$$\frac{\partial n}{\partial t} + \frac{v_0}{2}\frac{\partial n}{\partial x} - \frac{\partial}{\partial x}D_{xx}\frac{\partial n}{\partial x} = 0, \tag{30}$$

with the non-linear spatial diffusion coefficient given by

$$D_{xx} = \frac{1}{4v_0}\int_{v_{\min}}^{v_0}\frac{v^2\,(v_0 - v)^2}{D}\mathrm{d}v = \frac{v_0^2 n_p}{4\pi\omega_{pe} n(x,t)}\left(\ln\frac{v_0}{v_{\min}} - 1\right) \propto \frac{v_0^2}{4}\tau_{\mathrm{q}}\,. \tag{31}$$

The spatial diffusion coefficient (31) is dependent on the beam density and is smaller for smaller quasilinear time. In other words, the stronger the beam (larger n_b), the slower the diffusion term (smaller D_{xx}). The diffusion coefficient with $1/n(x,t)$ nonlinearity is often named *fast diffusion* i.e. diffusion is fast in regions where the density of particles is low (e.g. Juan R. Esteban & Vázquez 1988a; Vázquez 2017).

An important peculiarity of the solution (31) is that we had to integrate from $v_{\min}$ and when $v_{\min} \to 0$, $D_{xx} \to \infty$ diverges. The divergence has a physical reason: the plateau down to zero velocity is formed at $t \to \infty$, so the diffusion coefficient is infinite when $v_{\min} \to 0$. In other words, the spatial diffusion coefficient, $D_{xx} \propto \tau_{\mathrm{q}}$, is infinite due to the infinite time required to form plateau down to $v_{\min} = 0$. While the plateau is quickly formed over broad range of velocities (over quasilinear time $\tau_{\rm q}$), the growth rate of Langmuir waves is actually zero at $v=0$, as one can see from equation (2). Indeed, numerical simulations (e.g. Kontar et al. 1998; Kontar 2001b) show that the relaxation proceeds down to a small but finite velocity. In plasma with a Maxwellian distribution of thermal particles, the plateau is also formed between maximum beam velocity and thermal distribution, so that $v_{\rm min} \simeq 3-4v_{T_e}$ (e.g. Kontar & Pécseli 2002; Ziebell et al. 2008, 2011; Sauer et al. 2019).

Therefore, in order to compare our analytical model to the results of the numerical simulations or observations, we include a constant lower bound $v_{\rm min}$ to the plateau in velocity, resulting in the new electron distribution function

$$f^0\left(v,x,t\right) = \begin{cases} p\left(x,t\right), & v_{\rm min}<v<u(x,t) \\ 0, & v\leq v_{\rm min},\; v\geq u(x,t) \end{cases} \tag{32}$$

and the spectral energy density of Langmuir waves

$$W^0\left(v,x,t\right) = \begin{cases} W_0\left(v,x,t\right), & v_{\rm min}<v<u(x,t) \\ 0, & v\leq v_{\rm min},\; v\geq u(x,t) \end{cases} \tag{33}$$

where the solution for $u(x,t)$, $p(x,t)$, and $W_0(x,v,t)$ can be found following Kontar et al. (1998) to be

$$u(x,t) = v_0, \tag{34}$$

$$p(x,t) = \frac{n_b}{v_0-v_{\rm min}}\exp\left[-\frac{\left(x-\left(v_0+v_{\rm min}\right)t/2\right)^2}{d^2}\right], \tag{35}$$

$$W_0\left(v,x,t\right) = \frac{m}{\omega_{pe}}v^3\left(v-v_{\rm min}\right)\left(1-\frac{v+v_{\rm min}}{v_0+v_{\rm min}}\right)p(x,t), \tag{36}$$

The obvious difference from the solution (13-15) is that the solution (34-36) accounts for the minimum velocity of a plateau. Another consequence of $v_{\rm min}$ is that electron density is now given by

$$n(x,t) = \int\limits_{v_{\rm min}}^{v_0} p\left(x,t\right){\rm d}v = p\left(x,t\right)\left(v_0-v_{\rm min}\right), \tag{37}$$

with a new diffusion equation

$$\frac{\partial n}{\partial t}+\frac{v_0+v_{\rm min}}{2}\frac{\partial n}{\partial x}+\frac{\partial}{\partial x}\int\limits_{v_{\rm min}}^{v_0} vf^1{\rm d}v = 0\;. \tag{38}$$

After finding f^1 (see Appendix (B)), we arrive to

$$\int\limits_{v_{\rm min}}^{v_0} vf^1{\rm d}v = -\frac{1}{4(v_0-v_{\rm min})}\frac{\partial n}{\partial x}\int\limits_{v_{\rm min}}^{v_0}\frac{(v-v_{\rm min})^2\left(v_0-v\right)^2}{D}{\rm d}v\;. \tag{39}$$

Similarly to the previous subsection, the expression for D can be found from the formula for the spectral energy density $W_0(x,v,t)$ and the plateau height $p(x,t)$. From Kontar et al. (1998), we have that in case of non-zero $v_{\rm min}$

$$D = D_0v^2\left(v-v_{\rm min}\right)\left(1-\frac{v+v_{\rm min}}{v_0+v_{\rm min}}\right), \tag{40}$$

where now $D_0 = \pi\frac{\omega_{pe}}{(v_0-v_{\rm min})}\frac{n(x,t)}{n_p}$.

If we insert this D into equation (39), we obtain (see Appendix B)

$$\int\limits_{v_{\rm min}}^{v_0} vf_1{\rm d}v = -\frac{v_0+v_{\rm min}}{4D_0}\left(\frac{v_0+v_{\rm min}}{v_0-v_{\rm min}}\ln\frac{v_0}{v_{\rm min}}-2\right)\frac{\partial n}{\partial x}, \tag{41}$$

and our advection-nonlinear-diffusion becomes

$$\frac{\partial n}{\partial t}+\frac{(v_0+v_{\rm min})}{2}\frac{\partial n}{\partial x}-\frac{\partial}{\partial x}D_{xx}\frac{\partial n}{\partial x} = 0, \tag{42}$$

where our new diffusion coefficient D_{xx} that includes $v_{\min}$ is now given by

$$D_{xx} = \frac{v_0^2 - v_{\min}^2}{4\pi\omega_{pe}} \frac{n_p}{n(x,t)} \left(\left(\frac{v_0 + v_{\min}}{v_0 - v_{\min}} \right) \ln \frac{v_0}{v_{\min}} - 2 \right) = \frac{v_0^2 - v_{\min}^2}{4\pi} \tau_q \left(\left(\frac{v_0 + v_{\min}}{v_0 - v_{\min}} \right) \ln \frac{v_0}{v_{\min}} - 2 \right) . \tag{43}$$

The spatial diffusion coefficient D_{xx} is inversely proportional to electron number density $n(x,t)$ or is proportional to quasilinear time, so the spatial diffusion is faster for longer quasilinear time $n_p/(\omega_{pe} n(x,t))$. The diffusion coefficient D_{xx} is zero when $v_{\min} = v_0$, i.e. spatial diffusion is not possible without quasilinear relaxation. The electron beam diffusion coefficient (equation 43) is also dependent on $v_{\min}$, so the spatial expansion of electron beam is larger for smaller $v_{\min}$.

5. ASYMPTOTIC SOLUTION TO ADVECTION-NONLINEAR-DIFFUSION EQUATION

Let us consider the evolution of an electron beam given by initial condition

$$n(x, t = 0) = n_b \delta\left(x/d\right) , \tag{44}$$

where n_b is the electron beam density and d is the characteristic size. The advection-nonlinear diffusion equation (42) with $n(x,t)$ normalised with n_b can be rewritten

$$\frac{\partial n}{\partial t} + \frac{v_0 + v_{min}}{2} \frac{\partial n}{\partial x} - \frac{\partial}{\partial x} D_{xx}^0 \frac{n_b}{n} \frac{\partial n}{\partial x} = 0 , \tag{45}$$

where the nonlinear dependency of D_{xx} on $n(x,t)$ is explicitly highlighted by introducing $D_{xx} = D_{xx}^0 \frac{n_b}{n}$. The equation (45) can be solved for constant v_0 and $v_{\min}$ to find asymptotic solution

$$\begin{aligned} n(x,t) &= \left(\frac{(x - (v_0 + v_{\min})t/2)^2}{2D_{xx}^0 n_b t} + \frac{2\pi^2}{n_b d^2} D_{xx}^0 t \right)^{-1} = \\ &= \frac{n_b}{\pi} \frac{2\pi D_{xx}^0 t/d^2}{(x - (v_0 + v_{\min})t/2)^2/d^2 + 4\pi^2 (D_{xx}^0 t)^2/d^4}, \end{aligned} \tag{46}$$

which is a Lorentzian

$$L(x') = \frac{1}{\pi} \frac{\gamma}{x'^2 + \gamma^2} , \tag{47}$$

where $x' = (x - (v_0 + v_{\min})t/2)/d$ and $\gamma = 2\pi D_{xx}^0 t/d^2$. The solution (46) describes the expanding electron beam moving with the speed $(v_0 + v_{\min})/2$. The electron beam size given by γ is proportional to time t, and for $t \to 0$, $n(x,t) \to n_b d\delta(x)$, which is the initial condition (44). The electron density equation (45) without advection term has been studied for different applications (e.g. Lonngren & Hirose 1976; Berryman & Holland 1982; Juan R. Esteban & Vázquez 1988b; Hill & Hill 1993; King 1993; Rosenau 1995; Pedron et al. 2005) and the asymptotic profiles are often referred as ZKB profiles [from Zeldovich, Kompanyeets and Barenblat (see Barenblatt 1996, as a review)]. $n(x,t)$ given by (46) also conserves the number of particles $\int_{-\infty}^{+\infty} n(x,t) \mathrm{d}x = d n_b$ as expected.

The solution (46) shows that the electron beam always expands with time or with distance, so the peak of the beam at $x = (v_0 + v_{\min})t/2$ decreases following:

$$n\left(x, t = \frac{2x}{v_0 + v_{\min}}\right) = \frac{n_b d}{\pi} \frac{(v_0 + v_{\min})d}{4\pi x D_{xx}^0} \propto \frac{d}{x} , \tag{48}$$

which is an important result for the theory of type III bursts. The plausible decrease of electron number density with distance $1/x$ would be preferable to explain the intensity of type III burst with distance observed in the interplanetary space (e.g. Krupar et al. 2014).

Another interesting consequence of the solution (46) is that the beam size is growing with time or distance. Full Width at Half Maximum (FWHM) of the electron beam Δx is

$$\Delta x = 2\gamma d = \frac{4\pi}{d} D_{xx}^0 t = \frac{4\pi}{d} D_{xx}^0 \frac{2x}{(v_0 + v_{\min})} \propto \tau_{\mathrm{q}} \frac{x}{d} , \tag{49}$$

or dividing by the average speed of the electron beam $(v_0 + v_{\rm min})/2$, one obtains the time FWHM of the beam

$$\Delta t = \frac{4\gamma d}{v_0 + v_{\rm min}} = \frac{4\pi}{d} D^0_{xx} \frac{2t}{(v_0 + v_{\rm min})} = \frac{4\pi}{d} D^0_{xx} \frac{4x}{(v_0 + v_{\rm min})^2} \propto \tau_{\rm q} \frac{x}{d}\,, \tag{50}$$

where the spatial expansion of the beam Δx is linearly growing with time $\Delta x \propto t$ or the particle propagate ballistically, which is the special case of super-diffusion. Constant spatial diffusion coeficient leads to $\Delta x \propto t^{1/2}$, but the non-linear diffusion $D_{xx} \propto 1/n(x,t)$ due to Langmuir wave turbulence make electron beam to expand "faster", i.e. $\Delta x \propto t$ which is so-called super-diffusion (e.g. Okubo et al. 1984; Treumann 1997; Zimbardo et al. 2006). Here, the Langmuir turbulence is self-consistently generated as the electron beam propagates and expands in space. The level of Langmuir turbulence is proportional to the number of particles that gives the nonlinearity of the diffusion coefficient.

5.1. *Initially finite beam dynamics*

To compare with observations and numerical simulations, let us consider initial electron number density function as a Gaussian with characteristic size d, which is similar to the initial condition in (Kontar et al. 1998), i.e. the electron distribution function at $t = 0$ is

$$f(v, x, t = 0) = \frac{2n_b}{v_0} \frac{v}{v_0} \exp\left(-\frac{x^2}{d^2}\right)\,, \quad 0 < v < v_0\,, \tag{51}$$

hence the number density of beam electrons

$$n(x, t = 0) = n_b \exp\left(-x^2/d^2\right)\,, \tag{52}$$

with the total number of particles $\int_{-\infty}^{+\infty} n(x,t) = n_b d\sqrt{\pi}$. Then the solution of advection-nonlinear-diffusion equation (45) is the convolution of the initial condition (52) and Lorentzian from equation (46) normalised to 1, which is the solution to the Dirac delta function initial condition (the Green's function solution, which is an approximation when D_{xx} is nonlinear, see e.g. Kheifets (1984); Frasca (2008); Frank (2009)):

$$\begin{aligned} n(x,t) = \frac{n_b}{\pi} \int_{-\infty}^{\infty} \frac{2\pi D^0_{xx} t/d e^{-s^2/d^2} ds}{(x - s - \frac{v_0+v_{\rm min}}{2} t)^2 + 4\pi^2 (D^0_{xx} t)^2/d^2} = \\ = n_b \frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(\eta - y)^2 + \gamma^2} = n_b V(\gamma(t), \eta(x,t)), \end{aligned} \tag{53}$$

where $\eta(x,t) = (x - (v_0 + v_{\rm min})t/2)/d$, $\gamma(t) = 2\pi D^0_{xx} t/d^2$, $y = s/d$, and

$$V(\gamma, \eta) \equiv \frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{\gamma^2 + (\eta - y)^2}\,, \tag{54}$$

is the Voigt profile (Abramowitz & Stegun 1970), which is the convolution of Gaussian and Lorentzian, often used to fit spectral lines (e.g. Jeffrey et al. 2016).

In case of the solution (53), the width of the electron beam is the combination of Lorentzian FWHM given by equation (49) and the FWHM of the Gaussian (52), which is $\Delta x_G = 2\sqrt{\ln 2} d \simeq 1.67d$. The Voigt profile can be approximated (Whiting 1968):

$$\Delta x_{\rm V} \approx \Delta x/2 + \sqrt{\Delta x^2/4 + \Delta x_{\rm G}^2} \sim \sqrt{\Delta x_{\rm G}^2 + \Delta x^2}\,, \tag{55}$$

which shows that the electron beam of size d is expanding ballistically with time $\Delta x \propto t$, when $\Delta x \gg d$. The speed of the expansion (Equation 49) is controlled by the quasilinear time. Smaller/larger quasilinear time leads to slower/faster spatial electron beam expansion.

Figure 1 shows the spatial evolution of electron beam for the beam-plasma parameters used in the numerical simulations by Kontar et al. (1998). Unlike the solution assuming constant quasilinear time (Equation 14), the solution (Equation 53) is much closer in describing the simulated density profile showing both the decrease of the peak density and electron beam expansion (Figure 2 in Kontar et al. 1998).

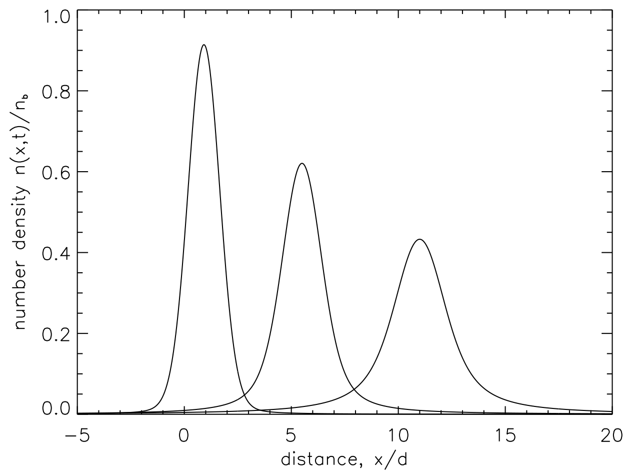

Figure 1. Electron number density profile $n(x,t)/n_b$ for the beam-plasma parameters as in the numerical simulations by Kontar et al. (1998): $n_b = 12$ cm^{-3}, $n_p = 6\times10^8$ cm^{-3} (i.e. $f_{pe} \simeq 220$ MHz) and $v_0 = 10^{10}$ cm/s, $v_{\min} = 0.1v_0$, $d = 3\times10^9$ cm. The three curves are the density profiles given by the solution (Equation 53) for $t = 0.5, 3, 6$ seconds.

Using the solution for electron beam spread (53), the spectral energy density of the Langmuir waves (Equation 36) becomes

$$W = n(x,t)\frac{m}{\omega_p}v^3\frac{v-v_{\min}}{v_0-v_{\min}}\left(1-\frac{v+v_{\min}}{v_0+v_{\min}}\right)\,, \tag{56}$$

so the spectral energy density decreases with distance due to the spatial evolution of $n(x,t)$.

The peak density of electrons $n(x, t = 2x/(v_0+v_{\min}))$ decreases with distance following

$$\frac{n\left(x, t=\frac{2x}{v_0+v_{\min}}\right)}{n_b} = V\left(\gamma\left(t=\frac{2x}{v_0+v_{\min}}\right), \eta=0\right)\,, \tag{57}$$

where $\gamma(x, t = 2x/(v_0+v_{\min})) = 4\pi D^0_{xx}x/((v_0+v_{\min})d^2)$. Figure 2 shows the peak value (Equation 57) as a function of distance.

The width variation or the time required to pass a specific point in space for a beam would correspond to the duration of type III burst at a given frequency. Interestingly, type III observations, similar to the predictions of ballistic expansion (Figure 2), also show expansion (Figure 10 in Reid & Kontar 2018). The detailed comparison would require taking into account radio-wave propagation. The rate of expansion is dependent on density and can be a new valuable diagnostic of electron beam density in type III bursts. This is also apparent from Figures 3 and 4, where the temporal evolution of simulated electron distribution, spectral energy density and electron beam density is shown for $n_b = 12$ cm^{-3} and $n_b = 120$ cm^{-3}. The value of $v_{\min}$ was chosen to be the minimum velocity value at half maximum of the electron distribution. It is evident that higher densities correspond to shorter quasilinear times of interaction, resulting in a better fit between simulations and the analytical solution.

6. SUMMARY

We develop a quantitative analytical model of the electron transport responsible for type III solar radio bursts. The developed model takes into account the finite size of electron beam, so the generation of Langmuir waves and quasilinear relaxation proceeds faster in the regions of higher electron number density. In the limit of small quasilinear time, the hydrodynamic approach yields the advection-nonlinear-diffusion equation for electron number density. Since the rate of relaxation of electrons is governed by the beam density at different spatial locations, the non-linear diffusion coefficient is inversely proportional to the beam density $D_{xx} \propto 1/n(x,t)$, process known as fast diffusion. Low electron beam density away from the peak of the electron beam leads to faster spatial diffusion of electrons.

The model has an elegant analytical solution showing that electron beam propagates at constant speed $(v_0+v_{\min})/2$ but with varying spatial width. The electron beam spatial size growths with the rate dependent on quasilinear time $\tau_{\rm q}$. The spatial width of the electron beam is proportional to $\tau_{\rm q}$ and to time t at large $x \gg d$. Unlike a linear diffusion

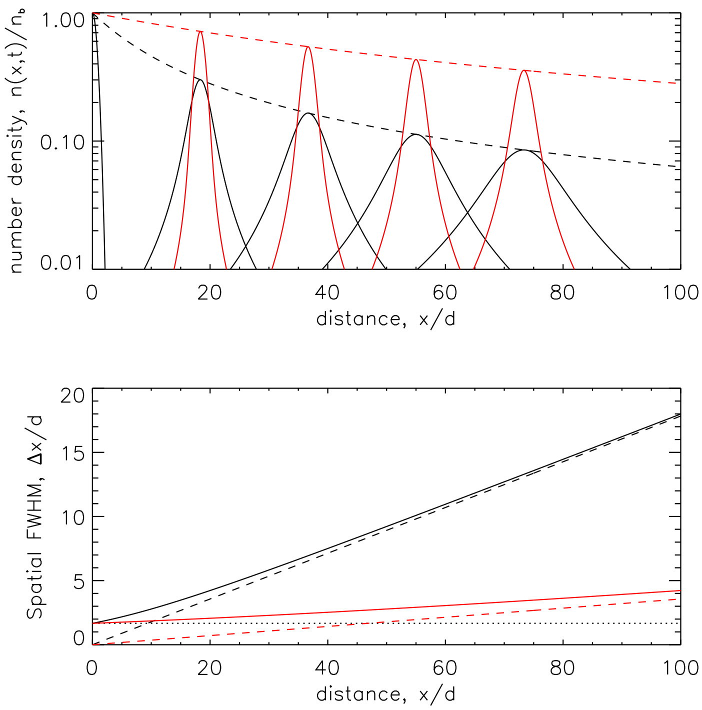

Figure 2. *Top panel:* Electron number density profile $n(x,t)/n_b$ at $t = 0, 20, 40, 60$ seconds for the same beam plasma parameters as in Figure 1 $n_b = 12$ cm^{-3} in black and $n_b = 60$ cm^{-3} in red . *Bottom panel:* FWHM width of the beam given by equation 55 (solid line). The dashed line is the width of Lorentzian (Equation 49). The horizontal dashed line is Gaussian FWHM, $\Delta x_G \simeq 1.67d$. Black lines are for $n_b = 12$ cm^{-3} and red lines are for $n_b = 60$ cm^{-3}.

case, when the beam size increases as $\propto \sqrt{t}$, the nonlinear diffusion leads to ballistic (super-diffusion) expansion i.e. electron beam size is $\propto t$ at large distances $x \gg d$ (see lower panel in Figure 2 and equation 49). Although the spatial expansion is linear with time, the rate of the expansion could be small for small quasilinear times $\tau_{\rm q}$ or large densities (compare Figures 4 and 3).

The spatial expansion of the electron beam leads to the decrease of the peak density of the electron beam. For large $x \gg d$, when the expansion is $\propto x$, the maximum beam density decreases as $\propto 1/x$, with the rate dependent on the beam density. The spectral energy density of Langmuir waves as the proxy for the type III solar radio flux is also decreasing $\propto 1/x$.

We further note that the spatial distribution of electrons has quantitative agreement with the numerical solutions of kinetic equations, where both numerical solutions and analytical density profiles show Voigt-like profile (Figure 1). Similar to the simulations, the peak density of electrons in the beam decreases with distance at the rate similar to the numerical solution (Figure 2 in Kontar et al. (1998)). The analytical solution also shows that on top of advection with constant speed, $(v_0 + u_{\rm min})/2$, the nonlinear diffusion leads to spatial expansion of electron beam with time. The FWHM of the electron beam, in the analytical solution is shown to be expanding ballistically, i.e. $\Delta x \propto t$ for $\Delta x \gg d$. The expansion of the electron beam is faster further away from the beam center due to larger local quasilinear time since $D_{xx} \propto 1/n(x,t)$ (Equation 49).

In application to type III solar radio bursts, the spectral energy density of plasma emission via Langmuir waves depends on the beam density and would decrease $\propto 1/x$, which is required to explain the radial type III solar burst flux variations (Krupar et al. 2014). The spatial expansion of the beam is also qualitatively better fit for the time width of type III bursts (Reid & Kontar 2018). However, more detailed studies including Langmuir wave refraction are likely to be required to have a detailed comparison with the solar type III burst observations.

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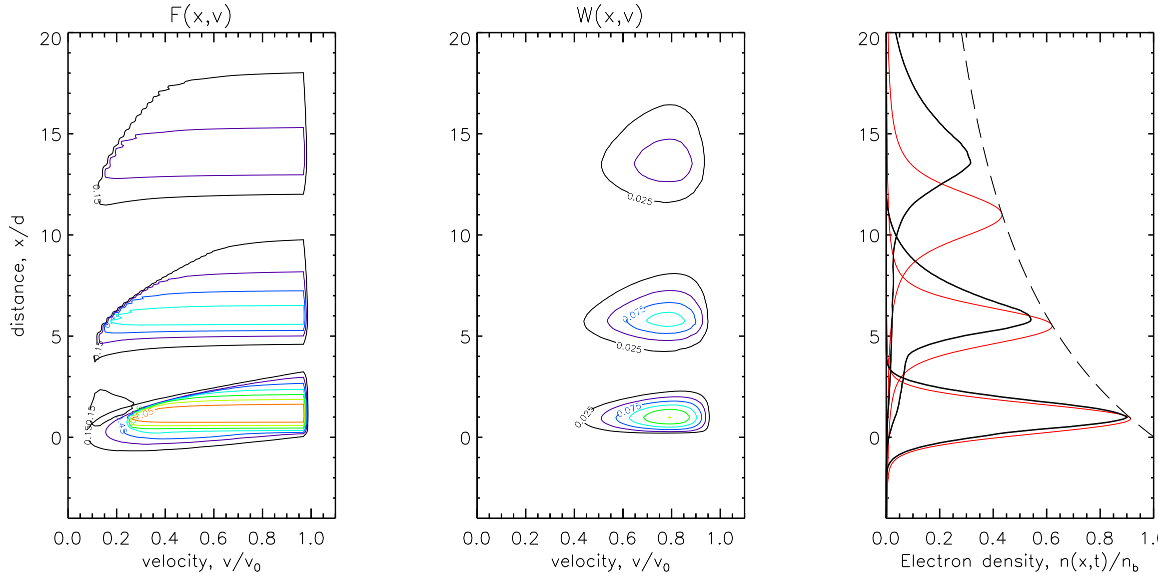

Figure 3. Simulated electron distribution $f(x,v,t)$ (left), spectral energy density $W(x,v,t)$ (center), and electron beam density $n(x,t)$ (right) at three time moments $t = 0.5, 3, 6$ s. for the following beam-plasma parameters $n_b = 12$ cm^{-3}, $n_p = 6\times10^8$ cm^{-3} (i.e. $f_{pe} \simeq 220$ MHz) and $v_0 = 10^{10}$ cm/s, $v_{\min} = 0.1v_0$, $d = 3\times10^9$ cm. The analytical density profile (53) is plotted in red, with the black dashed line showing its peak as a function of distance.

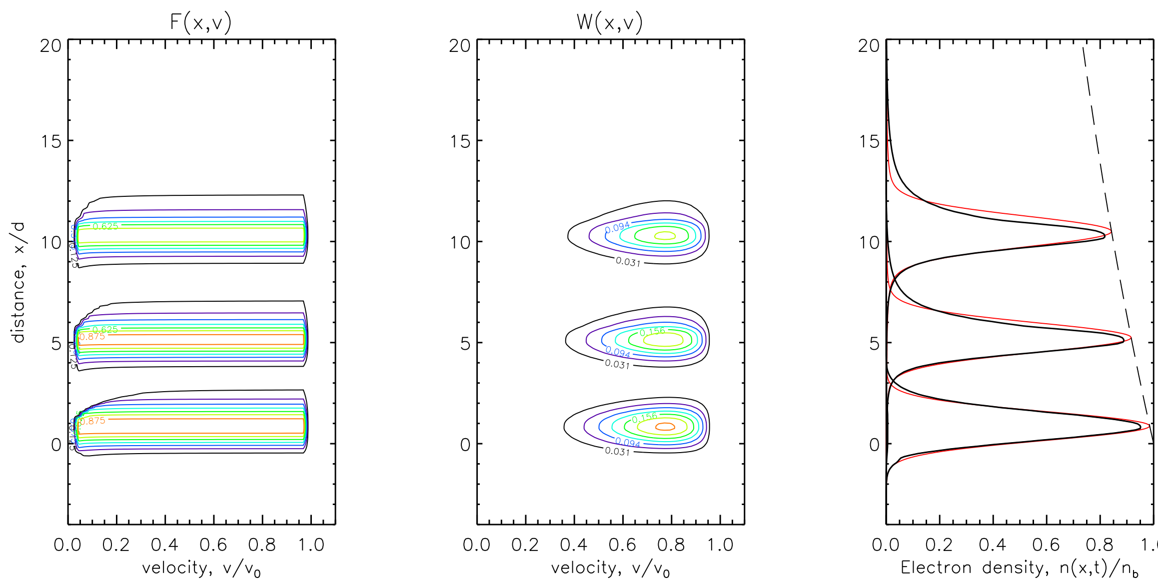

Figure 4. The same as Figure 3 but for $n_b = 120$ cm^{-3}.

APPENDIX

A. SPATIAL DIFFUSION COEFFICIENT

To determine the constant of integration C$_2$, we note that $\int_0^{v_0} f^1 dv = 0$, i.e.

$$\int_0^{v_0} f^1 dv = 0 = \left[vf^1\right]\Big|_0^{v_0} - \int_0^{v_0} v\frac{\partial f^1}{\partial v}dv\,,$$

which can be expanded into

$$\frac{1}{2v_0}\frac{\partial n}{\partial x}\int_0^{v_0}\left(v_0-v'\right)\frac{v'^2-v_0v'}{D}dv'+v_0C_2=0\,,$$

and rearranged to have

$$C_2=-\frac{1}{2v_0^2}\frac{\partial n}{\partial x}\int_0^{v_0}\left(v_0-v'\right)\frac{v'^2-v_0v'}{D}dv'\,. \tag{A1}$$

Substituting C_2 into equation (25), one obtains

$$f^1=\frac{1}{2v_0}\frac{\partial n}{\partial x}\left[\int_0^{v}\frac{v'^2-v_0v'}{D}dv'-\frac{1}{v_0}\int_0^{v_0}\left(v_0-v'\right)\frac{v'^2-v_0v'}{D}dv'\right]\,,$$

and taking into account that

$$\int_0^{v_0}vf^1dv=0=\left[\frac{v^2}{2}f^1\right]\Bigg|_0^{v_0}-\int_0^{v_0}\frac{v^2}{2}\frac{\partial f^1}{\partial v}dv\,,$$

one finds that

$$\int_0^{v_0}vf^1dv=-\frac{1}{4v_0}\frac{\partial n}{\partial x}\int_0^{v_0}v\left(v-v_0\right)\frac{v^2-v_0v}{D}dv=-\frac{1}{4v_0}\frac{\partial n}{\partial x}\int_0^{v_0}\frac{v^2\left(v_0-v\right)^2}{D}dv\,. \tag{A2}$$

B. SPATIAL DIFFUSION WITH NON-ZERO LOWER PLATEAU VELOCITY

Let us consider the plateau in the electron distribution function and spectral energy density including a constant lower bound $v_{\min}$:

$$f^0\left(v,x,t\right)=\begin{cases}p\left(x,t\right), & v_{\min}<v<u(x,t)\\ 0, & v\le v_{\min},\ v\ge u(x,t)\end{cases} \tag{B3}$$

$$W^0\left(v,x,t\right)=\begin{cases}W_0\left(v,x,t\right), & v_{\min}<v<u(x,t)\\ 0, & v\le v_{\min},\ v\ge u(x,t)\end{cases} \tag{B4}$$

Solutions for $u(x,t)$, $p(x,t)$, and $W_0(x,v,t)$ can be found following the method used in Kontar et al. (1998) to be

$$u(x,t)=v_0\,, \tag{B5}$$

$$p(x,t)=\frac{n_b}{v_0-v_{\min}}\exp\left[-\frac{\left(x-\left(v_0+v_{\min}\right)t/2\right)^2}{d^2}\right]\,, \tag{B6}$$

$$W_0\left(v,x,t\right)=\frac{m}{\omega_{pe}}v^3\left(v-v_{\min}\right)\left(1-\frac{v+v_{\min}}{v_0+v_{\min}}\right)p(x,t)\,, \tag{B7}$$

for the initial conditions on the electron distribution function given by equation (12).

Integrating equation (16) over velocity from $v_{\min}$ to ∞, one obtains

$$\frac{\partial n}{\partial t}+\frac{v_0+v_{\min}}{2}\frac{\partial n}{\partial x}+\frac{\partial}{\partial x}\int_{v_{\min}}^{v_0}vf^1\mathrm{d}v=0. \tag{B8}$$

Multiplying equation (16) by $v_0-v_{\min}$ and subtracting equation (B8) one finds

$$\left(v-\frac{v_0+v_{\min}}{2}\right)\frac{\partial n}{\partial x}+(v_0-v_{\min})\frac{\partial f^1}{\partial t}+(v_0-v_{\min})v\frac{\partial f^1}{\partial x}-\frac{\partial}{\partial x}\int_{v_{\min}}^{v_0}vf^1\mathrm{d}v=(v_0-v_{\min})\frac{\partial}{\partial v}D\frac{\partial f^1}{\partial v}. \tag{B9}$$

and retaining only zero order terms

$$\frac{1}{v_0-v_{\min}}\left(v-\frac{v_0+v_{\min}}{2}\right)\frac{\partial n}{\partial x}=\frac{\partial}{\partial v}D\frac{\partial f^1}{\partial v}\,, \tag{B10}$$

which is the equation for f^1. Integrating equation (B10) over v, one obtains

$$D\frac{\partial f^1}{\partial v} = \frac{1}{v_0 - v_{\min}}\frac{\partial n}{\partial x}\int\limits_{v_{\min}}^{v}\left(v^{'} - \frac{v_0 + v_{\min}}{2}\right)\mathrm{d}v^{'} = \frac{1}{v_0 - v_{\min}}\frac{\partial n}{\partial x}\frac{1}{2}\left(v - v_{\min}\right)\left(v - v_0\right) + C_1 \,. \tag{B11}$$

Since the quasilinear relaxation operates now on $v_{\min} < v < v_0$, we have $D|_{v=v_{\min}} = D|_{v=v_0} = 0$. Therefore, $C_1 = 0$, yielding

$$\frac{\partial f^1}{\partial v} = \frac{\left(v - v_{\min}\right)\left(v - v_0\right)}{2(v_0 - v_{\min})D}\frac{\partial n}{\partial x}\,, \tag{B12}$$

and with further integration over v, we find the expression for f^1

$$f^1 = \frac{1}{2(v_0 - v_{\min})}\frac{\partial n}{\partial x}\int\limits_{v_{\min}}^{v}\frac{(v^{'} - v_{\min})(v^{'} - v_0)}{D}\mathrm{d}v^{'} + C_2\,, \tag{B13}$$

where the constant C_2 is determined from

$$\int\limits_{v_{\min}}^{v_0} f^1\mathrm{d}v = 0 = \left[vf^1\right]\Big|_{v_{\min}}^{v_0} - \int\limits_{v_{\min}}^{v_0} v\frac{\partial f^1}{\partial v}\mathrm{d}v\ ,$$

which can be expanded into

$$\frac{1}{2(v_0 - v_{\min})}\frac{\partial n}{\partial x}\int\limits_{v_{\min}}^{v_0}(v_0 - v^{'})\frac{(v^{'} - v_{\min})(v^{'} - v_0)}{D}\mathrm{d}v^{'} + (v_0 - v_{\min})\,C_2 = 0.$$

Hence, the integration constant C_2 is found to be

$$C_2 = \frac{1}{2(v_0 - v_{\min})^2}\frac{\partial n}{\partial x}\int\limits_{v_{\min}}^{v_0}(v_0 - v^{'})^2\frac{(v^{'} - v_{\min})}{D}\mathrm{d}v^{'}\,, \tag{B14}$$

and f^1 can now be written as

$$f^1 = \frac{1}{2(v_0 - v_{\min})}\frac{\partial n}{\partial x}\left[\int\limits_{v_{\min}}^{v}\frac{(v^{'} - v_{\min})(v^{'} - v_0)}{D}\mathrm{d}v^{'} + \frac{1}{v_0 - v_{\min}}\int\limits_{v_{\min}}^{v_0}(v_0 - v^{\prime})^2\,\frac{(v^{'} - v_{\min})}{D}\mathrm{d}v^{\prime}\right]. \tag{B15}$$

This allows us to rewrite the integral in the diffusion term of equation (B8) as

$$\int\limits_{v_{\min}}^{v_0} vf^1\mathrm{d}v = \left[\frac{v^2}{2}f^1\right]_{v_{\min}}^{v_0} - \int\limits_{v_{\min}}^{v_0}\frac{v^2}{2}\frac{\partial f^1}{\partial v}\mathrm{d}v\ = -\frac{1}{4(v_0 - v_{\min})}\frac{\partial n}{\partial x}\int\limits_{v_{\min}}^{v_0}\frac{\left(v - v_{\min}\right)^2\left(v_0 - v\right)^2}{D}\mathrm{d}v. \tag{B16}$$

Using the diffusion coefficient in velocity space D from equation (40), we have the spatial diffusion coefficient

$$\begin{aligned} D_{xx} &= -\int\limits_{v_{\min}}^{v_0} vf^1\mathrm{d}v = \frac{1}{4(v_0 - v_{\min})}\int\limits_{v_{\min}}^{v_0}\frac{(v - v_{\min})^2(v_0 - v)^2}{D(v)}\mathrm{d}v \\ &= \frac{v_0 + v_{\min}}{4D_0(v_0 - v_{\min})}\int\limits_{v_{\min}}^{v_0}\frac{(v - v_{\min})(v_0 - v)}{v^2}\mathrm{d}v \\ &= \frac{v_0 + v_{\min}}{4D_0(v_0 - v_{\min})}\left[(v_0 + v_{\min})\ln(v) + \frac{v_0 v_{\min}}{v} - v\right]\Bigg|_{v_{\min}}^{v_0} \\ &= \frac{v_0 + v_{\min}}{4D_0}\left[\left(\frac{v_0 + v_{\min}}{v_0 - v_{\min}}\right)\ln\left(\frac{v_0}{v_{\min}}\right) - 2\right], \end{aligned} \tag{B17}$$

which is given in the main text by equation (43).

REFERENCES

Abramowitz, M., & Stegun, I. A. 1970, Handbook of mathematical functions : with formulas, graphs, and mathematical tables (U.S. Dept. of Commerce, National Bureau of Standards)

Akbari, H., LaBelle, J. W., & Newman, D. L. 2021, Frontiers in Astronomy and Space Sciences, 7, 116, doi: 10.3389/fspas.2020.617792

Bardwell, S., & Goldman, M. V. 1976, ApJ, 209, 912, doi: 10.1086/154790

Barenblatt, G. I. 1996, Scaling, Self-similarity, and Intermediate Asymptotics (Cambridge, UK: Cambridge University Press), doi: 10.1017/CBO9781107050242

Benz, A. O. 2017, Living Reviews in Solar Physics, 14, 2, doi: 10.1007/s41116-016-0004-3

Berryman, J. G., & Holland, C. J. 1982, Journal of Mathematical Physics, 23, 983, doi: 10.1063/1.525466

Chapman, S., & Cowling, T. G. 1970, The mathematical theory of non-uniform gases. an account of the kinetic theory of viscosity, thermal conduction and diffusion in gases (Cambridge: University Press)

Che, H., Goldstein, M. L., Diamond, P. H., & Sagdeev, R. Z. 2017, Proceedings of the National Academy of Science, 114, 1502, doi: 10.1073/pnas.1614055114

Drummond, W. E., & Pines, D. 1964, Annals of Physics, 28, 478, doi: 10.1016/0003-4916(64)90205-2

Fainberg, J., & Stone, R. G. 1970, SoPh, 15, 222, doi: 10.1007/BF00149487

Frank, T. D. 2009, Linear and Non-linear Fokker–Planck Equations (New York, NY: Springer New York), 5239–5265, doi: 10.1007/978-0-387-30440-3_311

Frasca, M. 2008, International Journal of Modern Physics A, 23, 299, doi: 10.1142/S0217751X08038160

Ginzburg, V. L., & Zhelezniakov, V. V. 1958, Soviet Ast., 2, 653

Goldman, M. V., & Dubois, D. F. 1982, Physics of Fluids, 25, 1062, doi: 10.1063/1.863839

Grognard, R. J. M. 1982, SoPh, 81, 173, doi: 10.1007/BF00151988

Hannah, I. G., Kontar, E. P., & Sirenko, O. K. 2009, ApJL, 707, L45, doi: 10.1088/0004-637X/707/1/L45

Hasselmann, K., & Wibberenz, G. 1970, ApJ, 162, 1049, doi: 10.1086/150736

Hill, D. L., & Hill, J. M. 1993, Quarterly of Applied Mathematics, 51, 633. http://www.jstor.org/stable/43637952

Holman, G. D., Aschwanden, M. J., Aurass, H., et al. 2011, SSRv, 159, 107, doi: 10.1007/s11214-010-9680-9

Jeffrey, N. L. S., Fletcher, L., & Labrosse, N. 2016, A&A, 590, A99, doi: 10.1051/0004-6361/201527986

Jokipii, J. R. 1966, ApJ, 146, 480, doi: 10.1086/148912

Juan R. Esteban, A. R., & Vázquez, J. L. 1988a, Communications in Partial Differential Equations, 13, 985, doi: 10.1080/03605308808820566

—. 1988b, Communications in Partial Differential Equations, 13, 985, doi: 10.1080/03605308808820566

Kaplan, S. A., & Tsytovich, V. N. 1973, Plasma astrophysics (Oxford: Pergamon Press)

Karlicky, M. 1997, SSRv, 81, 143, doi: 10.1023/A:1004939526282

Kheifets, S. 1984, Part. Accel., 15, 67

King, J. R. 1993, Philosophical Transactions of the Royal Society of London Series A, 343, 337, doi: 10.1098/rsta.1993.0052

Kontar, E. P. 2001a, A&A, 375, 629, doi: 10.1051/0004-6361:20010807

—. 2001b, Computer Physics Communications, 138, 222, doi: 10.1016/S0010-4655(01)00214-4

Kontar, E. P., Lapshin, V. I., & Melnik, V. N. 1998, Plasma Physics Reports, 24, 772

Kontar, E. P., & Pécseli, H. L. 2002, PhRvE, 65, 066408, doi: 10.1103/PhysRevE.65.066408

Krafft, C., & Savoini, P. 2023, ApJ, 949, 24, doi: 10.3847/1538-4357/acc1e4

Krucker, S., Kontar, E. P., Christe, S., & Lin, R. P. 2007, ApJL, 663, L109, doi: 10.1086/519373

Krupar, V., Maksimovic, M., Santolik, O., et al. 2014, SoPh, 289, 3121, doi: 10.1007/s11207-014-0522-x

Li, B., Cairns, I. H., & Robinson, P. A. 2008, Journal of Geophysical Research (Space Physics), 113, A06104, doi: 10.1029/2007JA012957

Lin, R. P. 1970, SoPh, 12, 266, doi: 10.1007/BF00227122

—. 1974, SSRv, 16, 189, doi: 10.1007/BF00240886

—. 1985, SoPh, 100, 537, doi: 10.1007/BF00158444

Lonngren, K. E., & Hirose, A. 1976, Physics Letters A, 59, 285, doi: 10.1016/0375-9601(76)90794-5

Lyubchyk, O., Kontar, E. P., Voitenko, Y. M., Bian, N. H., & Melrose, D. B. 2017, SoPh, 292, 117, doi: 10.1007/s11207-017-1140-1

Magelssen, G. R., & Smith, D. F. 1977, SoPh, 55, 211, doi: 10.1007/BF00150886

Mel'nik, V. N. 1995, Plasma Physics Reports, 21, 89, doi: 10.48550/arXiv.1802.07806

Mel'nik, V. N., & Kontar, E. P. 2000, NewA, 5, 35, doi: 10.1016/S1384-1076(00)00004-X

Mel'nik, V. N., Lapshin, V., & Kontar, E. 1999, SoPh, 184, 353, doi: 10.1023/A:1005191910544

Muschietti, L. 1990, SoPh, 130, 201, doi: 10.1007/BF00156790

Muschietti, L., Goldman, M. V., & Newman, D. 1985, SoPh, 96, 181, doi: 10.1007/BF00239800

Okubo, A., Mitchell, J., & Andreasen, V. 1984, Physics Letters A, 105, 169, doi: 10.1016/0375-9601(84)90389-X

Papadopoulos, K., Goldstein, M. L., & Smith, R. A. 1974, ApJ, 190, 175, doi: 10.1086/152862

Pedron, I. T., Mendes, R. S., Buratta, T. J., Malacarne, L. C., & Lenzi, E. K. 2005, PhRvE, 72, 031106, doi: 10.1103/PhysRevE.72.031106

Posner, A. 2007, Space Weather, 5, 05001, doi: 10.1029/2006SW000268

Ratcliffe, H., Kontar, E. P., & Reid, H. A. S. 2014, A&A, 572, A111, doi: 10.1051/0004-6361/201423731

Reid, H. A. S., & Kontar, E. P. 2013, SoPh, 285, 217, doi: 10.1007/s11207-012-0013-x

—. 2018, A&A, 614, A69, doi: 10.1051/0004-6361/201732298

Reid, H. A. S., Vilmer, N., & Kontar, E. P. 2014, A&A, 567, A85, doi: 10.1051/0004-6361/201321973

Rosenau, P. 1995, PhRvL, 74, 1056, doi: 10.1103/PhysRevLett.74.1056

Ryutov, D. D. 2018, Physics of Plasmas, 25, 100501, doi: 10.1063/1.5042254

Ryutov, D. D., & Sagdeev, R. Z. 1970, Soviet Journal of Experimental and Theoretical Physics, 31, 396

Sauer, K., Baumgärtel, K., Sydora, R., & Winterhalter, D. 2019, Journal of Geophysical Research (Space Physics), 124, 68, doi: 10.1029/2018JA025887

Schlickeiser, R. 1989, ApJ, 336, 243, doi: 10.1086/167009

Sturrock, P. A. 1964, NASA Special Publication, 50, 357

Takakura, T. 1982, SoPh, 78, 141, doi: 10.1007/BF00151150

Takakura, T., & Shibahashi, H. 1976, SoPh, 46, 323, doi: 10.1007/BF00149860

Timofeev, I. V., Annenkov, V. V., & Arzhannikov, A. V. 2015, Physics of Plasmas, 22, 113109, doi: 10.1063/1.4935890

Treumann, R. A. 1997, Geophys. Res. Lett., 24, 1727, doi: 10.1029/97GL01760

Vázquez, J. L. 2017, The Mathematical Theories of Diffusion: Nonlinear and Fractional Diffusion (Cham: Springer International Publishing), 205–278, doi: 10.1007/978-3-319-61494-6_5

Vedenov, A. A., Gordeev, A. V., & Rudakov, L. I. 1967, Plasma Physics, 9, 719, doi: 10.1088/0032-1028/9/6/305

Vedenov, A. A., & Velikhov, E. P. 1963, Soviet Journal of Experimental and Theoretical Physics, 16, 682

Vedenov, A. A., Velikhov, E. P., & Sagdeev, R. Z. 1961, Soviet Physics Uspekhi, 4, 332, doi: 10.1070/PU1961v004n02ABEH003341

Whiting, E. 1968, Journal of Quantitative Spectroscopy and Radiative Transfer, 8, 1379, doi: https://doi.org/10.1016/0022-4073(68)90081-2

Yoon, P. H., Ziebell, L. F., Gaelzer, R., Lin, R. P., & Wang, L. 2012, SSRv, 173, 459, doi: 10.1007/s11214-012-9867-3

Zaitsev, V. V., Mityakov, N. A., & Rapoport, V. O. 1972, SoPh, 24, 444, doi: 10.1007/BF00153387

Zheleznyakov, V. V., & Zaitsev, V. V. 1970, Soviet Ast., 14, 47

Ziebell, L. F., Gaelzer, R., & Yoon, P. H. 2008, Physics of Plasmas, 15, 032303, doi: 10.1063/1.2844740

Ziebell, L. F., Yoon, P. H., Pavan, J., & Gaelzer, R. 2011, Plasma Physics and Controlled Fusion, 53, 085004, doi: 10.1088/0741-3335/53/8/085004

Zimbardo, G., Pommois, P., & Veltri, P. 2006, ApJL, 639, L91, doi: 10.1086/502676